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On first order formulations of supergravities

Bernard Julia

Laboratoire de Physique Théorique CNRS-ENS
*24 rue Lhomond, F-75231 Paris Cédex 05, France**
E-mail: `caius@lpt.ens.fr`

Sebastian Silva

Max Planck Institut für Gravitationsphysik, Albert Einstein Institut
Am Mühlenberg 5, D-14476 Golm, Germany
E-mail: `Sebastian.Silva@lpt.ens.fr`

Abstract: Supergravities are usually presented in a so-called 1.5 order formulation. Here we present a general scheme to derive pure 1$^{\text{st}}$ order formulations of supergravities from the 1.5 order ones. The example of $\mathcal{N}_4 = 1$ supergravity will be rederived and new results for $\mathcal{N}_4 = 2$ and $\mathcal{N}_{11} = 1$ will be presented. It seems that beyond four dimensions the auxiliary fields introduced to obtain first order formulations of SUGRA theories do not admit supergeometrical transformation laws at least before a full superfield treatment. On the other hand first order formalisms simplify eventually symmetry analysis and the study of dimensional reductions.

Keywords: Field Theories in Higher Dimensions, Gauge Symmetry, Supergravity Models.

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Contents

1. Introduction

Since the discovery of the central importance of U-dualities to control the divergences of string theories [1, 2, 3] and of the duality between large-N super-Yang-Mills theory and AdS compactification of eleven dimensional supergravity [4] the need for a better conceptual understanding of the latter has become rather urgent. The superspace approach is notoriously hard but component approaches are rather cumbersome, this is unsatisfactory as more miracles are being discovered [5]. Finally the tensor calculus in 10 or 11 dimensions is strongly restricted by supersymmetry so it is important to streamline the corresponding constraints for instance on allowed counterterms and nonperturbative effects [6, 7].

The purpose of the present manuscript is to pursue the investigation of the general structure of supergravities in their first order formalism. In section 2, we will present a general method to derive first order formalisms. We will then treat the case of supergravities in general in section 3. The exemples of $\mathcal{N}_4 = 1$, $\mathcal{N}_4 = 2$ and $\mathcal{N}_{11} = 1$ will then illustrate the procedure, sections 4, 5 and 6.

A first order formalism for the three form of eleven dimensional supergravity was recently developped independently and with a different technique in [8], a first order formalism for the Lorentz connection in eleven dimensions had also been found in [9]. However our approach is both totally general and in the 11d case combines these two results in a nicely symmetrical fashion.

2. From 1.5 to 1st order formalisms

Supergravity theories are conveniently written in a 1.5 order formulation. It simply means that the lagrangean is a 1st order lagrangean, namely the spin connection is to be varied as an independent field, but on the other hand its local supersymmetry transformation law is not required and usually not known explicitly. The invariance of the action is only established after extremisation over the connection [10]. Both MacDowell and Mansouri [11] and Chamseddine and West [12] checked that the transformation laws of Deser and Zumino [13] are geometrical up to the variation of the Lorentz connection, they went on to notice that using the latter's equation of motion it does not matter, this remark applies equally well to the second order action of Freedman, van Nieuwenhuizen and Ferrara [14], this has been called 1.5 order formalism see for instance [10]. The group manifold approach in its original form or in the modified form of [15] does not suffice to obtain our complete results.

The purpose of this section is to show a simple way to construct, from a 1.5 formulation, a 1st order formulation, namely, to give a local supersymmetry transformation law for the spin connection, that leaves the lagrangean invariant.

Let $L(\varphi^a, \omega^i)$ be a first order lagrangean, which depends on some dynamical fields φ^a and on the auxiliary fields ω^i (the spin connection in the case of (super)gravity). The latter fields can be eliminated algebraically by making use of their equations of motion, that is:

$$E_i(\varphi, \omega) := \left.\frac{\partial L}{\partial \omega^i}\right|_{\varphi} = 0 \;\Rightarrow\; \omega^i = \omega_2^i(\varphi)\,. \tag{2.1}$$

We used the fact that the lagrangean can be written (up to a surface term) as an analytic function of ω^i but not of its derivatives.

Due to the analyticity of the lagrangean in the auxiliary fields, it is possible to perform a Taylor expansion around ω_2^i. Defining $\Delta\omega^i(\varphi, \omega) := \omega^i - \omega_2^i(\varphi)$, we obtain:[1]

$$L(\varphi^a, \omega^i) = L(\varphi^a, \omega_2^i) + \frac{\Delta\omega^i \Delta\omega^j}{2} X_{ij}(\varphi^a, \omega^i)\,, \tag{2.2}$$

[1]We will use the Einstein summation convention for the $_i$ indices (and *not* the De Witt one which includes spacetime integration).

where

$$X_{ij}(\varphi^a,\omega^i) := \sum_{n=0}^{\infty} \frac{2}{(2+n)!} \left. \frac{\partial^{2+n} L}{\partial\omega^i\partial\omega^j\partial\omega^{k_1}\cdots\partial\omega^{k_n}} \right|_{\omega=\omega_2} \Delta\omega^{k_1}\cdots\Delta\omega^{k_n} . \tag{2.3}$$

Note that the term linear in $\Delta\omega^i$ in the Taylor expansion (2.2) vanishes identically due to (2.1).

Let us suppose now that the action $\int L(\varphi^a,\omega_2^i)$ is invariant, as a 2$^{\rm nd}$ order functional, under a gauge symmetry, given by $\delta_2\varphi^a = \delta_2\varphi^a(\varphi)$. Our purpose will now be to show that it is possible to define a transformation law δ_1 that extends this gauge invariance to the 1$^{\rm st}$ order action $\delta_1 \int L(\varphi^a,\omega^i) = 0$. Let us first define $\delta_1\varphi^a$ for simplicity by:

$$\delta_1\varphi^a := \delta_2\varphi^a . \tag{2.4}$$

We need now a transformation law for the auxiliary fields $\delta_1\omega^i$ which, together with (2.4), leaves the 1$^{\rm st}$ order lagrangean invariant. To proceed, let us vary the equation (2.2) with respect to δ_1. Using equation (2.4) and the gauge invariance of the 2$^{\rm nd}$ order lagrangean under δ_2, we find that:

$$\delta_1 L(\varphi,\omega) = \delta_1\left(\frac{\Delta\omega^i\Delta\omega^j}{2}X_{ij}\right) = \Delta\omega^i X_{ij}\left(\delta_1\Delta\omega^j + \frac{1}{2}\Delta\omega^l \bar{X}^{jk}\delta_1 X_{kl}\right), \tag{2.5}$$

where $\bar{X}^{ij}$ is the inverse of the matrix X_{ij}.

Now, the 1$^{\rm st}$ order lagrangean will be invariant under the variation δ_1 if we impose the vanishing of (2.5) (up to some total derivative). This condition is implemented if the transformation law of the auxiliary fields is taken to be:

$$\delta_1\omega^i = \delta_2\omega_2^i - \frac{1}{2}\Delta\omega^k\bar{X}^{ij}\delta_1 X_{jk} \tag{2.6}$$

This formula shows that the (1$^{\rm st}$ order) transformation law for the auxiliary field ω^i is just the induced transformation law on ω_2^i, plus a term which vanishes modulo its equation of motion. This general result was presented in [16], without the explicit formula (2.6).

In general, X_{ij} depends on ω^i and some algebraic manipulations may be necessary to extract $\delta_1\omega^i$ from equation (2.6). In the simple cases where the lagrangean can be written (up to a surface term) as a polynomial of degree 2 in the auxiliary fields,[2] i.e. $L = \frac{1}{2}\omega^i\omega^j X_{ij}(\varphi) - \omega^i X_i(\varphi) + X_0(\varphi)$ (with $X_{ij}(\varphi)$, $X_i(\varphi)$ and $X_0(\varphi)$ being arbitrary functionals of the remaining dynamical fields and their derivatives), the above equation (2.6) reduces with the help of (2.4) to:

$$\delta_1\omega^i = \delta_2\omega_2^i(\varphi) + \frac{1}{2}\delta_2\bar{X}^{ij}(\varphi)E_j . \tag{2.7}$$

[2] This will be the case for all the applications treated here.

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This simple formula will be used in the following examples to find 1st order supergravities. It provides a first solution to our problem.

Sometimes however it is advisable to rewrite the transformation laws by making use of a twist, that is, a trivial gauge transformation in the terminology of [17] (see also [16]). In fact we can always redefine:

$$\bar{\delta}_1\varphi^a := \delta_1\varphi^a + \Omega^{ai}E_i(\varphi,\omega)\,, \tag{2.8}$$

$$\bar{\delta}_1\omega^i := \delta_1\omega^i - \Omega^{ai}E_a(\varphi,\omega)\,, \tag{2.9}$$

or/and

$$\bar{\delta}_1\omega^i = \delta_1\omega^i + \Xi^{ij}E_j \tag{2.10}$$

and similarly for the φ^a's.

Then, $\bar{\delta}_1$ is still a gauge symmetry of the 1st order lagrangean (for any Ω^{ai} and any antisymmetric matrix $\Xi^{ij} = \Xi^{[ij]}$). Note that the trivial gauge transformations (2.8)–(2.10) are just the simplest examples (which will be enough for our present purpose). In fact, these formulas remain unchanged in the De Witt notation, ie with a spacetime integration in addition to the Einstein summation, see for example [17, 18].

This kind of twist will be used to simplify the form of $\delta_1\varphi^a$. The examples of $\mathcal{N}_4 = 1$, $\mathcal{N}_4 = 2$ and $\mathcal{N}_{11} = 1$ supergravities are treated in the following sections. Note that due to the symmetry of the matrix $\bar{X}^{ij}$ (see the definition (2.3)), the last term of equation (2.7) cannot be twisted away.

We would like to conclude this section by the following observation: associated to a 2nd order formulation, there are an infinite number of corresponding 1.5 order theories. This is due to the fact that the choice of the auxiliary field is somewhat arbitrary. In fact, it is straightforward to verify that a redefinition of the type $\tilde{\omega}^i := \omega^i + f^i(\varphi)$, where $f^i(\varphi)$ is an arbitrary function of the fields φ^a and eventually of their derivatives, does not change the associated 2nd order theory. In the case of supergravities, there is a natural choice for the auxiliary gravitational connection which is the supercovariant ("hatted") connection (see below the origin of supercovariance in 1.5 formalism). The same is true for the supercovariant "hatted" field strength, if any, as we shall see below.

3. The case of supergravities

The purpose of this section is to apply the previous formulas to derive 1st order formulations for *all* supergravities from their 1.5 formulations. We indeed give a general formula for the supersymmetry variation of the connection for *any* supergravity in *any* dimension. Explicit calculations will then be given for the special cases of $\mathcal{N}_4 = 1$, $\mathcal{N}_4 = 2$ and $\mathcal{N}_{11} = 1$ supergravities in the next sections.

3.1 Generalities on supergravities

The bosonic fields of supergravities are generically the one-form vierbein $\boldsymbol{e}^a$ and eventually some (non)-abelian p-forms $\boldsymbol{A}$'s. The auxiliary fields associated to $\boldsymbol{e}^a$ and to the $\boldsymbol{A}$'s are respectively the one-form spin-connection $\boldsymbol{\omega}^a{}_b$ and the $(p+1)$-form field strength $\boldsymbol{F}$. Our aim is thus to compute the supersymmetry transformation laws for $\boldsymbol{\omega}^a{}_b$ and $\boldsymbol{F}$ using formula (2.7). This equation shows that the relevant part of the supergravity lagrangean is the one quadratic in the auxiliary fields. For *any* supergravity this is:

$$\boldsymbol{L}_{\rm sug}(\boldsymbol{\varphi},\boldsymbol{\omega},\boldsymbol{F}) = -\frac{1}{4\kappa^2}\boldsymbol{R}^{ab}\wedge\boldsymbol{\Sigma}_{ab} - \frac{1}{2}\boldsymbol{F}\wedge *\boldsymbol{F} + \text{other}\,, \tag{3.1}$$

where $\boldsymbol{R}^{ab} := \boldsymbol{d}\boldsymbol{\omega}^{ab} + \boldsymbol{\omega}^a{}_c\wedge\boldsymbol{\omega}^{cb}$ and $\boldsymbol{\Sigma}_{a_1\ldots a_r} := \frac{1}{(D-r)!}\varepsilon_{a_1\ldots a_r a_{r+1}\ldots a_D}\boldsymbol{e}^{a_{r+1}}\wedge\ldots\wedge\boldsymbol{e}^{a_D}$ (see also [19] for notation and conventions). By *other* we mean the rest of our supergravity action which is at most linear in the spin connection and in the field strength. Here $\boldsymbol{\varphi}$ denotes all the remaining fields, i.e. those of the second order formalism.

As a consequence of equation (3.1), the Euler-Lagrange equations associated to the spin-connection can always be written as:

$$\frac{\delta\boldsymbol{L}_{\rm sug}}{\delta\boldsymbol{\omega}^{ab}} = -\frac{1}{2\kappa^2}\Delta\boldsymbol{\omega}^c{}_{[a}\wedge\boldsymbol{\Sigma}_{b]c} = 0\,, \tag{3.2}$$

where $\Delta\boldsymbol{\omega}^a{}_b = \boldsymbol{\omega}^a{}_b - \boldsymbol{\omega}^a_{2b}$, $\boldsymbol{\omega}^a_{2b}(\boldsymbol{\varphi})$ being not an independent field but the unique connection which satisfies the equations of motion (3.2).

As the connection is an auxiliary field, the first equation (3.2) can be algebraically inverted for $D \geq 3$. In components, we find that:

$$\Delta\omega_{\mu ab} = M^{\nu cd}_{\mu ab}\frac{\delta L_{\rm sug}}{\delta\omega^{\nu cd}}\,,$$

where

$$M^{\nu cd}_{\mu ab} = 2\kappa^2\left(e^\nu_a e^{[c}_\mu\delta^{d]}_b - e^\nu_b e^{[c}_\mu\delta^{d]}_a - \delta^\nu_\mu\delta^{[c}_b\delta^{d]}_a + \frac{2}{D-2}e^{\nu[c}\left(\delta^{d]}_a e_{\mu b} - \delta^{d]}_b e_{\mu a}\right)\right). \tag{3.3}$$

3.2 The supersymmetry transformation law of the connection

The $2^{\rm nd}$ order supergravity lagrangean $\boldsymbol{L}_{\rm sug}(\boldsymbol{\varphi},\boldsymbol{\omega}_2,\boldsymbol{F}_2)$ is invariant under some local supersymmetry defined by its action on the non-auxiliary fields namely $\delta_2\boldsymbol{\varphi}$. We derive now the supersymmetry transformation law of the spin-connection for the $1^{\rm st}$ order lagrangean. The case of the field strength will be treated in the next subsection.

The method presented in the previous section to compute $\delta_1\boldsymbol{\omega}^a{}_b$ from (2.7) can be summarized as the following recipe: "keep the part of the lagrangean quadratic in the connection (namely the auxiliary field). Replace $\boldsymbol{\omega}^a{}_b$ by the difference $\Delta\boldsymbol{\omega}^a{}_b$ (defined after equation (3.2)) in this piece of the lagrangean. Then, impose the vanishing of its variation under δ_1". From (3.1), the only part of the supergravity lagrangean which is proportional to the square of the connection $\boldsymbol{\omega}^a{}_b$ comes from the purely gravitational

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Einstein-Hilbert piece, namely $\boldsymbol{\omega}^{a}{}_{c}\wedge\boldsymbol{\omega}^{cb}\wedge\boldsymbol{\Sigma}_{ab}$. Thus the previous recipe gives:

$$\delta_1\left(\Delta\boldsymbol{\omega}^{a}{}_{c}\wedge\Delta\boldsymbol{\omega}^{cb}\wedge\boldsymbol{\Sigma}_{ab}\right)=0\,, \tag{3.4}$$

and then, using assumption (2.4),

$$\delta_1\boldsymbol{\omega}^{c}{}_{[a}\wedge\boldsymbol{\Sigma}_{b]c}=\delta_2\boldsymbol{\omega}^{c}_{2[a}(\varphi)\wedge\boldsymbol{\Sigma}_{b]c}-\frac{1}{2}\Delta\boldsymbol{\omega}^{c}{}_{[a}\wedge\delta_2\boldsymbol{\Sigma}_{b]c}\,. \tag{3.5}$$

Note that $\delta_1\boldsymbol{\omega}^{a}{}_{b}$ can be extracted from (3.5) using the matrix M given in (3.3). The result is:

$$\delta_1\boldsymbol{\omega}_{ab}=\delta_2\boldsymbol{\omega}_{2ab}+\frac{1}{2}\left(T_{a,bc}-T_{c,ab}+T_{b,ca}\right)\boldsymbol{e}^{c}\,, \tag{3.6}$$

where we defined $T_{c,ab}:=\frac{2}{D-2}\eta_{c[a}\tilde{T}^{d}{}_{b]d}+\tilde{T}_{cab}$ with $\tilde{T}^{\mu}{}_{ab}:=\delta_2\boldsymbol{e}^{d}_{\rho}\Delta\omega_{\nu}{}^{c}{}_{[a}\delta^{\mu\nu\rho}_{b]cd}$.

Of course, the above result can also be obtained straightforwardly from equation (2.7).

3.3 The supersymmetry transformation law for the field strength

If the supergravity lagrangean depends also on some (non)-abelian p-form $\boldsymbol{A}$, we may wish to give a supersymmetry transformation law for the associated auxiliary field, namely the field strength $\boldsymbol{F}$. Using equation (3.1) in (2.5), this transformation law follows from (see also the recipe given in the previous example):

$$-\frac{1}{2}\delta_1\left(\Delta\boldsymbol{F}\wedge*\Delta\boldsymbol{F}\right)=0\,, \tag{3.7}$$

with $\Delta\boldsymbol{F}=\boldsymbol{F}-\boldsymbol{F}_2$, $\boldsymbol{F}_2(\boldsymbol{\varphi})$ being the induced field strength constructed from the dynamical fields $\boldsymbol{\varphi}$ (for instance $\boldsymbol{F}_2=\boldsymbol{dA}+\cdots$) through the equation of motion: $\delta\boldsymbol{L}_{\rm sug}/\delta\boldsymbol{F}|_{\boldsymbol{F}=\boldsymbol{F}_2}=0$.

Equation (3.7) gives the transformation law for $\boldsymbol{F}$:

$$\delta_1\boldsymbol{F}=\delta_2\boldsymbol{F}_2+\delta_2\boldsymbol{e}^{a}\wedge\boldsymbol{i}_a\Delta\boldsymbol{F}-\frac{\delta_2 e}{2e}\Delta\boldsymbol{F}\,, \tag{3.8}$$

where $\boldsymbol{i}_a$ denotes the interior product with respect to the vector field e^{μ}_{a} (the inverse of the vierbein form $\boldsymbol{e}^{a}=e^{a}_{\mu}\boldsymbol{d}x^{\mu}$) and e the determinant of the vierbein.

To summarize: a $1^{\rm st}$ order formulation of any supergravity with a 1.5 order action is given by the transformation laws (2.4), (3.6) and (3.8).

3.4 Some useful trivial transformations

It may be preferable to rewrite the transformation laws in a simpler form by twisting them slightly. The transformation law for the gravitino can be generically written as:

$$\delta_1\bar{\boldsymbol{\psi}}:=\boldsymbol{D}_2\bar{\epsilon}+\text{more}\,, \tag{3.9}$$

where $\boldsymbol{D}_2$ is the covariant derivative constructed with the induced connection $\boldsymbol{\omega}^{ab}_2$ and *more* is the remaining part of the supersymmetry transformation law, which does not contain any derivative of the transformation parameter $\bar{\epsilon}$.

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Let us then define a new transformation law for the gravitino (and for the connection), which differs from (3.9) by a term proportional to the equations of motion of $\boldsymbol{\omega}^a{}_b$ (3.2) (this makes the Lorentz covariance more obvious with $\boldsymbol{D}_2 \to \boldsymbol{D}$, $\boldsymbol{D}$ being the covariant derivative defined by the connection $\boldsymbol{\omega}^a{}_b$):

$$\bar{\delta}_1 \bar{\boldsymbol{\psi}} := \delta_1 \bar{\boldsymbol{\psi}} - \bar{\epsilon} \Delta \boldsymbol{\omega}^{ab} \frac{\gamma_{ab}}{4} = \boldsymbol{D}\bar{\epsilon} + \text{more} \tag{3.10}$$

$$\bar{\delta}_1 \boldsymbol{\omega}^{ab} := \delta_1 \boldsymbol{\omega}^{ab} + \boldsymbol{\Omega}^{ab} \tag{3.11}$$

and $\bar{\delta}_1$(other fields) $:= \delta_1$(other fields).

Here $\boldsymbol{\Omega}^{ab}$ is the one form given as in equation (2.9). A direct way to compute it explicitly is the following: $\boldsymbol{\Omega}^{ab}$ has to be such that $\boldsymbol{\Omega}^{ab} \wedge \delta \boldsymbol{L}_{\text{sug}} / \delta \boldsymbol{\omega}^{ab} - \bar{\epsilon} \Delta \boldsymbol{\omega}^{ab} \frac{\gamma_{ab}}{4} \wedge \delta \boldsymbol{L}_{\text{sug}} / \delta \bar{\boldsymbol{\psi}} = 0$. Using the equation of motion of the connection (3.2), we find that:

$$\boldsymbol{\Omega}^c{}_{[a} \wedge \boldsymbol{\Sigma}_{b]c} = -\frac{\kappa^2}{2} \bar{\epsilon} \gamma_{ab} \frac{\delta \boldsymbol{L}_{\text{sug}}}{\delta \bar{\boldsymbol{\psi}}} . \tag{3.12}$$

This kind of twist may sometimes significantly simplify the final expression for the connection transformation law.

Finally, if the transformation law of the gravitino (3.9) depends on the induced field strength $\boldsymbol{F}_2$, it is again possible to twist it in a similar way by adding some trivial gauge transformation proportional to the equation of motion of the field strength. The net result would be the change $\boldsymbol{F}_2 \to \boldsymbol{F}$ in (3.9).

4. The 1^{st} order $\mathcal{N}_4 = 1$ supergravity

As a first example let us now derive (systematically) the old result of Deser and Zumino [13] for the $\mathcal{N}_4 = 1$ supergravity in 4 dimensions. Let us first use the general method of the previous section. That is, starting from a 1.5 formulation we will derive the connection supersymmetry transformation law and improve it by adding a trivial gauge transformation. In the last subsection we shall present a new method to obtain the same result in a more straightforward way. We shall use this method in the next sections to treat both the $\mathcal{N}_4 = 2$ and the $\mathcal{N}_{11} = 1$ supergravities.

4.1 The lagrangean and the equations of motion

Let us start with the 1^{st} order lagrangean of $\mathcal{N}_4 = 1$ supergravity[3] (as given by the so-called 1.5 formalism):

$$\boldsymbol{L}_1 = -\frac{1}{4\kappa^2} \boldsymbol{R}^{ab} \wedge \boldsymbol{\Sigma}_{ab} + \frac{i}{2} \bar{\boldsymbol{\psi}} \wedge \gamma^5 \boldsymbol{\gamma}_{(1)} \wedge \boldsymbol{D} \boldsymbol{\psi} , \tag{4.1}$$

[3]The conventions are the following (see also appendix A): $\eta^{ab} = \{-,+,+,+\}$, $\epsilon_{0123} = 1$, γ^a are four real Majorana matrices, $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, $\left(\gamma^5\right)^2 = -\mathbb{1}$, $4\kappa^2 = 16\pi G$, G being the Newton constant.

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where $\boldsymbol{\gamma}_{(1)} := \gamma_a \boldsymbol{e}^a$, $\boldsymbol{D\psi} := \boldsymbol{d\psi} + \frac{\gamma_{ab}}{4}\boldsymbol{\omega}^{ab}\wedge\boldsymbol{\psi}$ with $\gamma_{a_1 \ldots a_r} := \gamma_{[a_1} \cdots \gamma_{a_r]}$ and the definitions of $\boldsymbol{R}^{ab}$ and $\boldsymbol{\Sigma}_{ab}$ were given in the previous section. The equations of motion corresponding to lagrangean (4.1) are:

$$\frac{\delta \boldsymbol{L}_1}{\delta \boldsymbol{e}^a} = -\frac{1}{4\kappa^2}\boldsymbol{R}^{bc}\wedge\boldsymbol{\Sigma}_{bca} - \frac{i}{2}\bar{\boldsymbol{\psi}}\gamma^5\gamma_a\wedge\boldsymbol{D\psi} = 0 \tag{4.2}$$

$$\frac{\delta \boldsymbol{L}_1}{\delta \bar{\boldsymbol{\psi}}} = i\gamma^5\boldsymbol{\gamma}_{(1)}\wedge\boldsymbol{D\psi} - \frac{i}{2}\gamma^5\gamma_a\left(\boldsymbol{\Theta}^a - \frac{i\kappa^2}{2}\bar{\boldsymbol{\psi}}\gamma^a\wedge\boldsymbol{\psi}\right)\wedge\boldsymbol{\psi} = 0 \tag{4.3}$$

$$\frac{\delta \boldsymbol{L}_1}{\delta \boldsymbol{\omega}^{ab}} = -\frac{1}{4\kappa^2}\left(\boldsymbol{\Theta}^c - \frac{i\kappa^2}{2}\bar{\boldsymbol{\psi}}\gamma^c\wedge\boldsymbol{\psi}\right)\wedge\boldsymbol{\Sigma}_{abc} = 0 \tag{4.4}$$

with $\boldsymbol{\Theta}^a := \boldsymbol{de}^a + \boldsymbol{\omega}^a{}_b\wedge\boldsymbol{e}^b$.

Note that the very last term of equation (4.3), namely $-\frac{\kappa^2}{4}\gamma^5\gamma_a\boldsymbol{\psi}\wedge(\bar{\boldsymbol{\psi}}\gamma^a\wedge\boldsymbol{\psi})$, vanishes identically after a Fierz rearrangement. It has been however added to (4.3) to emphasize the vanishing of the sum of the last two terms after making use of the equations of motion of the connection (4.4).

4.2 The supersymmetry transformation law

The lagrangean (4.1) is invariant under local supersymmetry as a 2$^{\text{nd}}$ order functional of $\boldsymbol{e}^a$ and $\boldsymbol{\psi}$. That is, if we use the equations of motion of the connection $\boldsymbol{\omega}^{ab}$, the following transformation laws leave the action invariant:

$$\delta_2\bar{\boldsymbol{\psi}} = \boldsymbol{D}_2\bar{\epsilon} = \boldsymbol{d}\bar{\epsilon} - \bar{\epsilon}\boldsymbol{\omega}_2^{ab}(\boldsymbol{e},\boldsymbol{\psi})\frac{\gamma_{ab}}{4}\,. \tag{4.5}$$

$$\delta_2\boldsymbol{e}^a = i\kappa^2\bar{\epsilon}\gamma^a\boldsymbol{\psi}\,, \tag{4.6}$$

A purely 1$^{\text{st}}$ order symmetry of supergravity is defined by $\delta_1\boldsymbol{e}^a := \delta_2\boldsymbol{e}^a$ and $\delta_1\bar{\boldsymbol{\psi}} := \delta_2\bar{\boldsymbol{\psi}}$, plus the supersymmetry transformation law for the connection given by (3.5). Using the fact that $\boldsymbol{\omega}_2^{ab}$ identically satisfies equation (4.4), we can compute $\delta_1\boldsymbol{\omega}_2^{ab}(\boldsymbol{e}^a,\boldsymbol{\psi})$ in a simple way. Equation (3.5) becomes after some algebra:

$$\delta_1\boldsymbol{\omega}^c{}_{[a}\wedge\boldsymbol{\Sigma}_{b]c} = -\frac{i\kappa^2}{2}\bar{\epsilon}\gamma^c\boldsymbol{D\psi}\wedge\boldsymbol{\Sigma}_{abc} - \boldsymbol{A}_{ab} + \boldsymbol{B}_{ab} + \boldsymbol{C}_{ab}\,, \tag{4.7}$$

where we defined

$$\boldsymbol{A}_{ab} := \frac{1}{4}\Delta\boldsymbol{\omega}^c{}_d\wedge\boldsymbol{e}^d\wedge\delta_2\boldsymbol{\Sigma}_{abc}\,, \tag{4.8}$$

$$\boldsymbol{B}_{ab} := \frac{i\kappa^2}{2}\bar{\epsilon}\gamma^5\gamma_{[a}\Delta\boldsymbol{\omega}_{b]c}\wedge\boldsymbol{e}^c\wedge\boldsymbol{\psi}\,, \tag{4.9}$$

$$\boldsymbol{C}_{ab} := \frac{i\kappa^2}{4}\bar{\epsilon}\gamma^5\boldsymbol{\gamma}\wedge\Delta\boldsymbol{\omega}_{ab}\,. \tag{4.10}$$

Note that we just found an explicit expression for $\delta_1\boldsymbol{\omega}^c{}_{[a}\wedge\boldsymbol{\Sigma}_{b]c}$, which can (always) be inverted (as equation (3.2)) in any spacetime dimension $D \geq 3$ using the matrix (3.3). Finally, we see from (4.7) that there is no term proportional to the derivative of the gauge parameter (namely $\boldsymbol{d}\bar{\epsilon}$) in the transformation law of the connection (in other words the connection remains supercovariant). This is due to two facts:

- The *induced* connection $\boldsymbol{\omega}_2^{ab}(\boldsymbol{e},\boldsymbol{\psi})$ was in this case effortlessly supercovariant, its supercovariance is still obscure though and it may seem that it is a four-dimensional accident [10]; more precisely it is only in four dimensions that one may require simultaneously the independence of the connection and the absence of fermionic quartic terms in the action, in fact here we shall always use the supercovariant connection as the independent variable by using the freedom mentioned at the end of section 2.

- The last term of the r.h.s. of (3.5) (or of (3.6)) is also supercovariant since it depends on $\bar{\epsilon}$ only through $\delta_2\boldsymbol{e}^a$ (which is independent of $\partial_\mu\bar{\epsilon}$, see (4.6)).

We may pause at this stage to reflect on the general structure of supersymmetry transformation laws. The anticommutator of two of them gives a diffeomorphism consequently one has the typical multiplet structure (B,F,A) with B bosonic, F fermionic and A auxiliary ie nonpropagating fields. Their variations are schematically

$$\begin{aligned} &\delta B \propto F\,, \\ &\delta F \propto A + \partial B\,, \\ &\delta A \propto \partial F\,. \end{aligned} \tag{4.11}$$

If one insists on a geometrical approach one is led to replace the term ∂B by the Lorentz connection in the case of the graviton multiplet but this uses the equation of motion for the connection so first order formalisms and offshell supersymmetry algebra are somewhat antinomic. (W. Siegel informed us recently that he has made progress on this issue).

Nevertheless supercovariance is useful to compute the superpotential corresponding to the supersymmetry gauge invariance in 1^{st} order formalism, see [20].

At this point the definition of $\delta_1\boldsymbol{\omega}^{ab}$ given by (4.7), together with (4.5)–(4.6) provide a 1^{st} order formulation of supergravity. It is however preferable to add two trivial gauge transformations to simplify the previous expressions:

1. The last term of equation (4.7), namely $\boldsymbol{C}_{ab}$ (4.10), can be eliminated. In fact it corresponds to a trivial gauge transformation of the type (2.10). This can be checked as follows: due to equation (3.2), we have that $\delta_1\boldsymbol{\omega}^{ab}\wedge\delta\boldsymbol{L}_1/\delta\boldsymbol{\omega}^{ab} = -\frac{1}{2\kappa^2}\Delta\boldsymbol{\omega}^{ab}\wedge\delta_1\boldsymbol{\omega}^c{}_a\wedge\boldsymbol{\Sigma}_{bc}$. Then, $\boldsymbol{C}_{ab}$ contributes with a term proportional to $\Delta\boldsymbol{\omega}^{ab}\wedge\Delta\boldsymbol{\omega}_{ab}$, which vanishes identically (the connection is a one-form).

2. The second trivial gauge transformation was explained in section 3.4 through equations (3.9)–(3.12). It allows to rewrite the gravitino transformation law in terms of the 1st order covariant derivative $\boldsymbol{D}$ (instead of $\boldsymbol{D}_2$):

$$\bar{\delta}_1\bar{\boldsymbol{\psi}} := \boldsymbol{D}\bar{\epsilon}\,, \tag{4.12}$$

$$\bar{\delta}_1\boldsymbol{\omega}^{ab} := \delta_1\boldsymbol{\omega}^{ab} + \boldsymbol{\Omega}^{ab}\,. \tag{4.13}$$

The one-form $\boldsymbol{\Omega}^{ab}$ is computed from (3.12) using the gravitino equation of motion (4.3). After some rearrangement, we find that:

$$\boldsymbol{\Omega}^c{}_{[a}\wedge\boldsymbol{\Sigma}_{b]c} = -i\kappa^2\bar{\epsilon}\gamma^5\gamma_{[a}\boldsymbol{e}_{b]}\wedge\boldsymbol{D\psi} + \frac{i\kappa^2}{2}\bar{\epsilon}\gamma^c\boldsymbol{D\psi}\wedge\boldsymbol{\Sigma}_{abc} + \boldsymbol{A}_{ab} - \boldsymbol{B}_{ab}\,, \tag{4.14}$$

where $\boldsymbol{A}_{ab}$ and $\boldsymbol{B}_{ab}$ were respectively defined in (4.8) and (4.9).

The twisted 1st order supersymmetric variation of the connection is then given simply by combining (3.11) together with (4.7) and (4.14):

$$\bar{\delta}_1\boldsymbol{\omega}^c{}_{[a}\wedge\boldsymbol{\Sigma}_{b]c} = -i\kappa^2\bar{\epsilon}\gamma^5\gamma_{[a}\boldsymbol{e}_{b]}\wedge\boldsymbol{D\psi}\,. \tag{4.15}$$

Note that all the terms proportional to the equations of motion of the connection (through $\Delta\boldsymbol{\omega}^{ab}$), namely $\boldsymbol{A}_{ab}$ and $\boldsymbol{B}_{ab}$ disappear.

One finally recovers the old result of Deser and Zumino [13] after inverting (4.15) using the matrix (3.3) (as for (3.2)):

$$\bar{\delta}_1\boldsymbol{\omega}_{ab} = 2i\kappa^2\bar{\epsilon}\gamma^5\left(\boldsymbol{\gamma}_{(1)}{*}\psi_{ab} - \frac{1}{2}\boldsymbol{e}_a\gamma^c{*}\psi_{cb} + \frac{1}{2}\boldsymbol{e}_b\gamma^c{*}\psi_{ca}\right), \tag{4.16}$$

where ${*}\psi_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D^\rho\psi^\sigma$.

Thus, the first order formulation of $\mathcal{N}_4 = 1$ supergravity is given by the lagrangean (4.1) and the locally supersymmetric transformation laws (4.6), (4.12) and (4.16) (or in a less aesthetical way by (4.5), (4.6) and (4.7)). But the relative simplicity of (4.16) (comparable to that of $\delta_2\boldsymbol{\omega}_2^{ab}(\boldsymbol{e},\boldsymbol{\psi})$) remains mysterious.

4.3 A new method

Our idea now is to formulate better the choice of the twist (trivial gauge transformation) and then to rewrite the equation for $\bar{\delta}_1\boldsymbol{\omega}^{ab}$. Geometrically, one would like to keep (4.6) and (4.12) and find the $\bar{\delta}_1\boldsymbol{\omega}^{ab}$ of (4.16) in a more straightforward and less surprising way.

Let us begin with a general discussion: suppose that our lagrangean is invariant under a gauge symmetry given by:

$$\delta_\xi\boldsymbol{\phi}^i = \boldsymbol{d}\xi^a\wedge\boldsymbol{\Delta}_a^i + \xi^a\tilde{\boldsymbol{\Delta}}_a^i\,, \tag{4.17}$$

where $\boldsymbol{\phi}^i$ is now a p_i-form which goes for all the fields (even auxiliary) of the theory.

Moreover, we suppose that the theory is a first order one, in the sense that $\delta_\xi \boldsymbol{\phi}^i$ and $\boldsymbol{E}_i := \delta \boldsymbol{L}/\delta\boldsymbol{\phi}^i$ depend on $\boldsymbol{\phi}^i$ and $\boldsymbol{d\phi}^i$ (and no more derivatives of the fields). Let us now define the $(D-1)$-form:

$$\boldsymbol{W}_\xi := \xi^a \boldsymbol{\Delta}^i_a \wedge \boldsymbol{E}_i \,. \tag{4.18}$$

Then, the following identity holds (offshell of course):

$$\frac{\delta \boldsymbol{W}_\xi}{\delta \boldsymbol{\phi}^i} = \frac{\partial}{\partial \boldsymbol{d\phi}^i}\left(\delta_\xi \boldsymbol{\phi}^j \wedge \boldsymbol{E}_j\right), \tag{4.19}$$

where $\delta/\delta\boldsymbol{\phi}^i = \partial/\partial\boldsymbol{\phi}^i - (-)^{p_i}\boldsymbol{d}\partial/\partial\boldsymbol{d\phi}^i$.

The proof is given in [21] (see also [18]). The point is now that the identity (4.19) can be used to compute straightforwardly $\bar{\delta}_1 \boldsymbol{\omega}^{ab}$ from $\bar{\delta}_1 \boldsymbol{e}^a$ and $\bar{\delta}_1 \boldsymbol{\psi}$. Before starting concrete calculations, we will comment on the general meaning of equation (4.19).

There is a fundamental reason why (4.19) is a useful formula for computing transformation laws. In fact, it is the lagrangean (covariant) version of a standard formula of hamiltonian formalism. Namely in the hamiltonian formalism, a transformation law $\delta_\xi \boldsymbol{\Phi}^I$ can be computed from the charge Q_ξ and the symplectic structure Ω_{IJ} by the following formula:

$$\delta_\xi \boldsymbol{\Phi}^I = \left\{Q_\xi, \boldsymbol{\Phi}^I\right\}_{PB}, \tag{4.20}$$

where now $\boldsymbol{\Phi}^I$ are phase space fields. The Poisson bracket (PB) is defined by the inverse of the symplectic structure by $\{,\} := (\delta/\delta\boldsymbol{\Phi}^I)\Omega^{IJ}(\delta/\delta\boldsymbol{\Phi}^J)$.

Then, equation (4.20) can be rewritten as:

$$\frac{\delta Q_\xi}{\delta \boldsymbol{\Phi}_I} = \delta_\xi \boldsymbol{\Phi}^J \Omega_{JI} \,. \tag{4.21}$$

Although (4.19) and (4.21) look different, they are in some way analogous. First of all the l.h.s. of (4.19) is nothing but the Euler-Lagrange variation of the Noether current $\boldsymbol{J}_\xi$ (and so the local covariant version of Q_ξ of (4.21)). This is a direct consequence of the fact that $\boldsymbol{J}_\xi = \boldsymbol{W}_\xi + \boldsymbol{dU}$ ($\boldsymbol{U}$ being the superpotential) [19]. Now, the r.h.s. of (4.19) contains two terms. The first one, $\delta_\xi\boldsymbol{\phi}^j \wedge (\partial/\partial\boldsymbol{d\phi}^i)\boldsymbol{E}_j$, corresponds to $\delta_\xi \boldsymbol{\Phi}^J\Omega_{JI}$ of (4.21). In fact, $(\partial/\partial\boldsymbol{d\phi}^i)\boldsymbol{E}_j$ is antisymmetric in i and j and is a local version (non integrated) of the Witten-Crnkovic-Zuckerman 2-form [22] (see [21, 18]) times $\delta_\xi\boldsymbol{\phi}^j$. The second term, namely $(\partial/\partial\boldsymbol{d\phi}^i)\delta_\xi\boldsymbol{\phi}^j\wedge\boldsymbol{E}_j$, vanishes on-shell. It is not so disturbing to find such an additional contribution in our covariant formulas since it also exists in the hamiltonian formalism; in fact, the equivalence between (4.21) and the symmetry of the original action is only up to some identifications of the type "$\dot{q} = p$". In our case, this kind of identifications are part of the equations of motion (1$^{\text{st}}$ order theories).

Let us come back to our $\mathcal{N}_4 = 1$ supergravity and use the identity (4.19) to compute the supersymmetry transformation law of the connection. First of all, we will comment on the following points derived in previous sections:

- By using some trivial gauge transformation, it is always possible to rewrite the supersymmetry transformation law of the gravitino in such a way that it depends on the independent field $\boldsymbol{\omega}^{ab}$ instead of the induced auxiliary field $\boldsymbol{\omega}_2^{ab}(\boldsymbol{\varphi})$. The mathematical consequence of this twist is that now $\partial\bar{\delta}_1\bar{\boldsymbol{\psi}}/\partial\partial_\mu e_\nu^a = 0$ (see the definition (4.12)). That was obviously false for $\delta_2\bar{\boldsymbol{\psi}}$ of (4.5).

- The induced connexion $\boldsymbol{\omega}_2^{ab}(\boldsymbol{\varphi})$ has been chosen to be supercovariant in the sense that its supersymmetry variation does not depend on the derivative of the gauge parameter $\bar{\epsilon}$. In that case, the independent connexion $\boldsymbol{\omega}^{ab}$ will also be supercovariant. This is a consequence of the general result (3.5)–(3.6) (itself following rather strong hypothesis on the gravitational kinetic term) and of the fact that both the supersymmetry transformation law of the vierbein (4.6) and the trivial gauge transformations (3.11)–(3.12) do not depend on the derivative of the gauge parameter $\partial_\mu\bar{\epsilon}$.

Two conclusions followed:

1. there should exist some $\bar{\delta}_1\boldsymbol{\omega}^{ab}$, which, together with $\bar{\delta}_1\bar{\boldsymbol{\psi}} = \boldsymbol{D}\bar{\epsilon}$ and $\bar{\delta}_1\boldsymbol{e}^a = i\kappa^2\bar{\epsilon}\gamma^a\boldsymbol{\psi}$, leave the supergravity action (4.1) invariant.
2. this $\bar{\delta}_1\boldsymbol{\omega}^{ab}$ does not depend on the derivatives of the parameter $\bar{\epsilon}$.

We are now ready to use the identity (4.19). From the previous discussion, the gravitino is the only field whose supersymmetry transformation law depends on the derivative of $\bar{\epsilon}$. Thus, the definitions (4.17) and (4.18) give:

$$\boldsymbol{W}_{\bar{\epsilon}} = \bar{\epsilon}\frac{\delta\boldsymbol{L}_1}{\delta\bar{\boldsymbol{\psi}}}\,. \tag{4.22}$$

Replacing the equation (4.3) in the above expression, we compute:

$$\frac{\delta\boldsymbol{W}_{\bar{\epsilon}}}{\delta\boldsymbol{e}^a} = \frac{i}{2}\bar{\epsilon}\gamma^5\gamma_a\boldsymbol{D}\boldsymbol{\psi} - \frac{i}{2}\boldsymbol{D}\bar{\epsilon}\wedge\gamma^5\gamma_a\boldsymbol{\psi}\,. \tag{4.23}$$

According to the identity (4.19), this expression is equal to

$$\frac{\partial}{\partial\boldsymbol{d}\boldsymbol{e}^a}\left(\bar{\delta}_1\boldsymbol{e}^b\wedge\frac{\delta\boldsymbol{L}_1}{\delta\boldsymbol{e}^b} + \bar{\delta}_1\bar{\boldsymbol{\psi}}\wedge\frac{\delta\boldsymbol{L}_1}{\delta\bar{\boldsymbol{\psi}}} + \bar{\delta}_1\boldsymbol{\omega}^{cd}\wedge\frac{\delta\boldsymbol{L}_1}{\delta\boldsymbol{\omega}^{cd}}\right). \tag{4.24}$$

Using the known expressions for $\bar{\delta}_1\boldsymbol{e}^b$ (4.6), $\bar{\delta}_1\bar{\boldsymbol{\psi}}$ (4.12) (which do not depend on the derivatives of the vierbein, see the first comment above) and the equations of motion (4.2)–(4.3), our expression becomes:

$$-\frac{i}{2}\boldsymbol{D}\bar{\epsilon}\wedge\gamma^5\gamma_a\boldsymbol{\psi} + \frac{\partial}{\partial\boldsymbol{d}\boldsymbol{e}^a}\left(\bar{\delta}_1\boldsymbol{\omega}^{cd}\wedge\frac{\delta\boldsymbol{L}_1}{\delta\boldsymbol{\omega}^{cd}}\right). \tag{4.25}$$

So equations (4.23) and (4.25) (together with (4.19)) imply:

$$\frac{\partial}{\partial \boldsymbol{d}\boldsymbol{e}^a}\left(\bar{\delta}_1\boldsymbol{\omega}^{cd}\wedge\frac{\delta \boldsymbol{L}_1}{\delta\boldsymbol{\omega}^{cd}}\right)=\frac{i}{2}\bar{\epsilon}\gamma^5\gamma_a\boldsymbol{D}\boldsymbol{\psi}\,. \tag{4.26}$$

This kind of equation will reappear in a more complicated form in the case of $\mathcal{N}_4 = 2$ or $\mathcal{N}_{11} = 1$ supergravities, that is why we shall now investigate it in general. Let us denote by ω^i the auxiliary fields, E_i their associated equations of motion and by x^α the $\boldsymbol{d}\boldsymbol{e}^a$'s (thus α stands for two spacetime indices and one Lorentz index). Then, equation (4.26) can be rewritten as:

$$\partial_\alpha\left(\bar{\delta}_1\omega^i E_i\right)=R_\alpha\,. \tag{4.27}$$

In our case, the equations E_i are linear in x^α (that is, the equations of the connection are linear in the derivatives of the vierbeins). The x^α dependence can be expressed as $E_i = E_{i\alpha}x^\alpha + E_{i0}$. On the other hand from (4.26) one reads that R_α is independent of x^α. In that case, we can *choose* $\bar{\delta}_1\omega^i$ to be also independent of x^α, and then (4.27) implies that:

$$\bar{\delta}_1\omega^i E_{i\alpha}=R_\alpha\,. \tag{4.28}$$

We used the word *choose* because (4.27) determines $\delta_1\omega^i$ only up to some trivial gauge symmetry of the type (2.10), which can eventually depend on x^α. The important point is that $\bar{\delta}_1\omega^i$ independent of x^α is consistent with (4.26). The last step is to invert equation (4.28) to extract $\bar{\delta}_1\omega^i$. In our case, this is possible because the indices i and α have the *same* dimension, namely $D^2(D-1)/2$. This is due to the fact that $\boldsymbol{\omega}^{ab}$ and $\boldsymbol{e}^a$ are, in a covariant sense, canonically conjugate fields.

Thus, using the first identity of (4.4), we can find a solution for $\bar{\delta}_1\boldsymbol{\omega}^{cd}$, from (4.28):

$$-\frac{1}{4\kappa^2}\bar{\delta}_1\boldsymbol{\omega}^{cd}\wedge\boldsymbol{\Sigma}_{acd}=\frac{i}{2}\bar{\epsilon}\gamma^5\gamma_a\boldsymbol{D}\boldsymbol{\psi}\,. \tag{4.29}$$

This last equation is equivalent to (4.15). This can be checked by "wedging" the above expression with $\boldsymbol{e}_b$ and antisymmetrising in the a and b indices. Then, it can be inverted to extract $\bar{\delta}_1\boldsymbol{\omega}^{ab}$. The result is of course again given by (4.16).

This new method will be used in the following sections to treat the $\mathcal{N}_4 = 2$ and $\mathcal{N}_{11} = 1$ supergravities.

5. 1st order formalism for $\mathcal{N}_4 = 2$ supergravity

Contrary to the $\mathcal{N}_4 = 1$ case, there seems to be no published first order formalism for $\mathcal{N}_4 = 2$ supergravity. We will give it here.

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5.1 The 1^{st} order lagrangean

The lagrangean of $\mathcal{N}_4 = 2$ supergravity in $D = 4$ dimensions depends on the one-form vierbein $\boldsymbol{e}^a$, the associated one-form so$(1,3)$ (spin) connection $\boldsymbol{\omega}^a{}_b$, two Rarita-Schwinger Majorana spinor-one-forms the gravitinos $\boldsymbol{\psi}^A$ ($A = 1, 2$ is the internal global SO(2) R-symmetry index) and one abelian one-form gauge connection, the photon $\boldsymbol{A}$, and its associated field strength $\boldsymbol{F}$. The 4-form lagrangean is given by[4] [10]:

$$\begin{aligned}\boldsymbol{L}_2 = &-\frac{1}{4\kappa^2}\boldsymbol{R}^{ab}\wedge\boldsymbol{\Sigma}_{ab} + \frac{i}{2}\bar{\boldsymbol{\psi}}_A\wedge\gamma^5\boldsymbol{\gamma}_{(1)}\wedge\boldsymbol{D}\boldsymbol{\psi}^A - \frac{1}{2}\boldsymbol{F}\wedge{*\boldsymbol{F}} + \\ &+ ({*\boldsymbol{F}} - \boldsymbol{b})\wedge(\boldsymbol{dA} - \boldsymbol{a}) - \frac{1}{2}\boldsymbol{a}\wedge\boldsymbol{b}\,, \end{aligned} \tag{5.1}$$

with the definitions

$$\boldsymbol{a} := \frac{i\kappa}{2}\varepsilon^A{}_B\bar{\boldsymbol{\psi}}_A\wedge\boldsymbol{\psi}^B \qquad \boldsymbol{b} := \frac{i\kappa}{2}\varepsilon^A{}_B\bar{\boldsymbol{\psi}}_A\gamma^5\wedge\boldsymbol{\psi}^B\,. \tag{5.2}$$

Notations, conventions and useful formulas can be found in appendix A. Moreover, the $*$ symbol refers to standard Hodge-duality.

The equations of motion of (5.1) are:

$$\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{e}^a} = -\frac{1}{4\kappa^2}\boldsymbol{R}^{bc}\wedge\boldsymbol{\Sigma}_{bca} - \frac{i}{2}\bar{\boldsymbol{\psi}}_A\gamma^5\gamma_a\wedge\boldsymbol{D}\boldsymbol{\psi}^A + \boldsymbol{\tau}_a = 0\,, \tag{5.3}$$

$$\begin{aligned}\frac{\delta\boldsymbol{L}_2}{\delta\bar{\boldsymbol{\psi}}_A} = &\, i\gamma^5\boldsymbol{\gamma}_{(1)}\wedge\hat{\boldsymbol{D}}\boldsymbol{\psi}^A - \frac{i}{2}\gamma^5\gamma_a\boldsymbol{\psi}^A\wedge\left(\boldsymbol{\Theta}^a - \frac{i\kappa^2}{2}\bar{\boldsymbol{\psi}}_B\gamma^a\wedge\boldsymbol{\psi}^B\right) + \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\boldsymbol{\psi}^B\wedge(\boldsymbol{F} - \boldsymbol{dA} + \boldsymbol{a}) = 0\,, \end{aligned} \tag{5.4}$$

$$\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{A}} = \boldsymbol{d}\,({*\boldsymbol{F}} - \boldsymbol{b}) = 0\,, \tag{5.5}$$

$$\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{\omega}^{ab}} = -\frac{1}{4\kappa^2}\left(\boldsymbol{\Theta}^c - \frac{i\kappa^2}{2}\bar{\boldsymbol{\psi}}_A\gamma^c\wedge\boldsymbol{\psi}^A\right)\wedge\boldsymbol{\Sigma}_{abc} = 0\,, \tag{5.6}$$

$$\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{F}} = *\,(\boldsymbol{dA} - \boldsymbol{a} - \boldsymbol{F}) = 0\,, \tag{5.7}$$

Let us comment the above equations:

- The last term in the r.h.s. of (5.3), namely $\boldsymbol{\tau}_a$, denotes the electromagnetic energy-momentum tensor and is given[5] by $\boldsymbol{\tau}_a = \frac{\partial}{\partial\boldsymbol{e}_a}({*\boldsymbol{F}}\wedge(\boldsymbol{dA} - \boldsymbol{a} - \frac{1}{2}\boldsymbol{F}))$.
- The supercovariant derivative appearing in the gravitini's equations of motion (5.4) is defined by: $\hat{\boldsymbol{D}}\boldsymbol{\psi}^A := \boldsymbol{D}\boldsymbol{\psi}^A - \frac{\kappa}{2}\varepsilon^A{}_B(\boldsymbol{F}_a\gamma^a + {*\boldsymbol{F}}_a\gamma^a\gamma^5)\wedge\boldsymbol{\psi}^B$, with $\boldsymbol{F}_a := \boldsymbol{i}_{\boldsymbol{e}_a}\boldsymbol{F} = F_{ab}\boldsymbol{e}^b$ and ${*\boldsymbol{F}}_a := \boldsymbol{i}_{\boldsymbol{e}_a}({*\boldsymbol{F}}) = ({*F})_{ab}\boldsymbol{e}^b$. To compute the result (5.4), the following two identities are useful:

[4] The conventions are the same as for the previous subsection (see also appendix A) together with the antisymmetric SO(2)-invariant tensor $\varepsilon^1{}_2 = -\varepsilon^2{}_1 = 1$ and $\varepsilon^A{}_C\varepsilon^C{}_B = -\delta^A_B$.

[5] A more explicit formula can be given for $\boldsymbol{\tau}_a$ by using that $\frac{\partial}{\partial\boldsymbol{e}^a}(\boldsymbol{F}\wedge{*\boldsymbol{G}}) = (-)^p\boldsymbol{F}\wedge\boldsymbol{i}_a({*\boldsymbol{G}}) - \boldsymbol{i}_a\boldsymbol{F}\wedge{*\boldsymbol{G}}$, for $\boldsymbol{F}$ and $\boldsymbol{G}$ any p-forms independent of $\boldsymbol{e}^a$, see [23, 18].

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(i) the Fierz rearrangement

$$\gamma_a \boldsymbol{\psi}^A \wedge (\bar{\boldsymbol{\psi}}_B \gamma^a \wedge \boldsymbol{\psi}^B) + \varepsilon^A{}_B \boldsymbol{\psi}^B \wedge (\varepsilon^C{}_D \bar{\boldsymbol{\psi}}_C \wedge \boldsymbol{\psi}^D) + \\ + \varepsilon^A{}_B \gamma^5 \boldsymbol{\psi}^B \wedge (\varepsilon^C{}_D \bar{\boldsymbol{\psi}}_C \gamma^5 \wedge \boldsymbol{\psi}^D) = 0\,,$$

(ii) a supercovariant derivative identity

$$i\gamma^5 \boldsymbol{\gamma}_{(1)} \wedge \hat{\boldsymbol{D}} \boldsymbol{\psi}^A = i\gamma^5 \boldsymbol{\gamma}_{(1)} \wedge \boldsymbol{D} \boldsymbol{\psi}^A - i\kappa \left(* \boldsymbol{F} + \gamma^5 \boldsymbol{F} \right) \wedge \varepsilon^A{}_B \boldsymbol{\psi}^B\,.$$

Note also that the last two terms of (5.4) vanish by making use of the auxiliary fields equations of motion (5.6)–(5.7).

- The unique solution to the field strength equation of motion, namely $\boldsymbol{F}_2 = \boldsymbol{dA} - \boldsymbol{a}$ is the so-called supercovariant field strength, in the sense that its induced supersymmetry transformation law (see equations (5.8)–(5.10) below) does not depend on the derivatives of the gauge parameter[6] $\bar{\epsilon}$. If we eliminate the auxiliary field $\boldsymbol{F}$, we recover the so-called "1st" order formulation, where the gravitational part of the lagrangean is really 1st order whereas the Maxwell part is not.

- The Bianchi identity and equation of motion for the $U(1)$ gauge field read symmetrically: $d(\boldsymbol{F}_2 + \boldsymbol{a}) = 0$ and $d(*\boldsymbol{F}_2 - \boldsymbol{b}) = 0$. The supercovariantisation term in the field strength is dual to the fermionic "source" in its equation of motion. Both quadratic expressions in the fermions should be rewritten (using the transformation rules given in the next section) as part of supercovariant differentiation.

5.2 The supersymmetry transformation laws

The 2nd order lagrangean which is obtained from (5.1) after eliminating the auxiliary fields $\boldsymbol{\omega}^{ab}$ and $\boldsymbol{F}$ is invariant under the following supersymmetry transformation laws:

$$\delta_2 \bar{\boldsymbol{\psi}}_A = \hat{\boldsymbol{D}}_2 \bar{\epsilon}_A := \boldsymbol{D}_2 \bar{\epsilon}_A - \frac{\kappa}{2} \varepsilon^B{}_A \bar{\epsilon}_B \left(\hat{\boldsymbol{F}}_{2a} \gamma^a - * \hat{\boldsymbol{F}}_{2a} \gamma^5 \gamma^a \right), \tag{5.8}$$

$$\delta_2 \boldsymbol{e}^a = i\kappa^2 \bar{\epsilon}_A \gamma^a \boldsymbol{\psi}^A\,, \tag{5.9}$$

$$\delta_2 \boldsymbol{A} = i\kappa \varepsilon^A{}_B \bar{\epsilon}_A \boldsymbol{\psi}^B\,. \tag{5.10}$$

As before, the index $_2$ in the r.h.s. of (5.8) indicates that the auxiliary fields are the *induced* ones, namely $\boldsymbol{D}_2 = \boldsymbol{D}(\boldsymbol{\omega}_2^{ab}(\boldsymbol{e}, \boldsymbol{\psi}))$ and $\boldsymbol{F}_2 = \boldsymbol{dA} - \boldsymbol{a}$.

The purpose is now to extend the 2nd order supersymmetry transformation laws (5.8)–(5.10) to the 1st order lagrangean (5.1). To proceed, we will use the new method presented in section 4.3. Let us first emphasize three points:

[6]The same happens for the induced spin connexion as in the $\mathcal{N}_4 = 1$ case, namely it is already supercovariant.

1. We can already give a 1st order formulation for the $\mathcal{N}_4 = 2$ supergravity by $\delta_1\bar{\boldsymbol{\psi}}_A = \delta_2\bar{\boldsymbol{\psi}}_A$, $\delta_1\boldsymbol{e}^a = \delta_2\boldsymbol{e}^a$, $\delta_1\boldsymbol{A} = \delta_2\boldsymbol{A}$ (equation (2.4)) and by equations (3.6) and (3.8) for the transformation law of the auxiliary fields.

2. For our choice of action the *induced* auxiliary fields $\boldsymbol{\omega}_2^{ab}$ and $\boldsymbol{F}_2$ are supercovariants, in the sense that their supersymmetry transformation laws (induced by (5.8)–(5.10)) do not contain derivatives of the gauge parameter (i.e. no terms proportional to $\partial_\mu\epsilon$).

3. There should exist some transformation laws $\bar{\delta}_1\boldsymbol{\omega}^{ab}$ and $\bar{\delta}_1\boldsymbol{F}$ which, together with

$$\begin{aligned}\bar{\delta}_1\bar{\boldsymbol{\psi}}_A &:= \hat{\boldsymbol{D}}\bar{\epsilon}_A\,,\\ \bar{\delta}_1\boldsymbol{e}^a &:= \delta_2\boldsymbol{e}^a\,,\\ \bar{\delta}_1\boldsymbol{A} &:= \delta_2\boldsymbol{A}\,,\end{aligned} \tag{5.11}$$

leave the lagrangean (5.1) invariant (up to some surface term). In fact, $\bar{\delta}_1$ differ from the δ_1 (see the above point 1.) by some trivial gauge transformations (which allow $\boldsymbol{\omega}_2^{ab} \to \boldsymbol{\omega}^{ab}$ and $\boldsymbol{F}_2 \to \boldsymbol{F}$ in (5.8)) of the type (2.8)–(2.9).

To compute simultaneously $\bar{\delta}_1\boldsymbol{\omega}^{ab}$ and $\bar{\delta}_1\boldsymbol{F}$, we will use the identity (4.19). First of all, these two transformation laws will differ from $\delta_2\boldsymbol{\omega}_2^{ab}$ and $\delta_2\boldsymbol{F}_2$ by some terms which are proportional to $\bar{\epsilon}$ (and which vanish on-shell). Thus the gravitini are the only fields whose supersymmetry transformation law contains terms proportional to $\partial_\mu\bar{\epsilon}$. In that case, the $(D-1)$-form defined by (4.18) becomes:

$$\boldsymbol{W}_\epsilon = \bar{\epsilon}_A\frac{\delta\boldsymbol{L}_2}{\delta\bar{\boldsymbol{\psi}}_A}\,. \tag{5.12}$$

Using then the first identity in (5.4), we can compute the following $(D-2)$-forms:

$$\frac{\delta\boldsymbol{W}_{\bar{\epsilon}}}{\delta\boldsymbol{e}^a} = \frac{i}{2}\bar{\epsilon}_A\gamma^5\gamma_a\boldsymbol{D}\boldsymbol{\psi}^A - \frac{i}{2}\boldsymbol{D}\bar{\epsilon}_A\wedge\gamma^5\gamma_a\boldsymbol{\psi}^A - i\kappa\frac{\partial{*\boldsymbol{F}}}{\partial\boldsymbol{e}^a}\wedge\bar{\epsilon}_A\varepsilon^A{}_B\boldsymbol{\psi}^B\,, \tag{5.13}$$

$$\frac{\delta\boldsymbol{W}_{\bar{\epsilon}}}{\delta\boldsymbol{A}} = -i\kappa\boldsymbol{d}\left(\bar{\epsilon}_A\varepsilon^A{}_B\gamma^5\boldsymbol{\psi}^B\right) = -i\kappa\boldsymbol{D}\left(\bar{\epsilon}_A\varepsilon^A{}_B\gamma^5\boldsymbol{\psi}^B\right). \tag{5.14}$$

Following the identity (4.19), these equations have to be equal respectively to $\frac{\partial}{\partial\boldsymbol{de}^a}\left(\bar{\delta}_1\boldsymbol{\phi}^i\wedge\delta\boldsymbol{L}_2/\delta\boldsymbol{\phi}^i\right)$ and to $\frac{\partial}{\partial\boldsymbol{dA}}\left(\bar{\delta}_1\boldsymbol{\phi}^i\wedge\delta\boldsymbol{L}_2/\delta\boldsymbol{\phi}^i\right)$, where $\boldsymbol{\phi}^i$ goes for all the fields (even auxiliary). Using now (5.3)–(5.5) together with (5.11), the pair of equations (5.13)–(5.14) has to be equal to:

$$\begin{aligned}(5.13) = &-\frac{i}{2}\bar{\delta}_1\bar{\boldsymbol{\psi}}_A\wedge\gamma^5\gamma_a\boldsymbol{\psi}^A + \bar{\delta}_1\boldsymbol{A}\wedge\frac{\partial\boldsymbol{d}{*\boldsymbol{F}}}{\partial\boldsymbol{de}^a} +\\ &+\frac{\partial}{\partial\boldsymbol{de}^a}\left(\bar{\delta}_1\boldsymbol{\omega}^{cd}\wedge\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{\omega}^{cd}} + \bar{\delta}_1\boldsymbol{F}\wedge\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{F}}\right),\end{aligned} \tag{5.15}$$

$$(5.14) = \bar{\delta}_1 \boldsymbol{e}^b \wedge \frac{\partial * \boldsymbol{F}}{\partial \boldsymbol{e}^b} - i\kappa \bar{\delta}_1 \bar{\boldsymbol{\psi}}_A \wedge \varepsilon^A{}_B \gamma^5 \boldsymbol{\psi}^B + \\ + \frac{\partial}{\partial \boldsymbol{dA}} \left(\bar{\delta}_1 \boldsymbol{\omega}^{cd} \wedge \frac{\delta \boldsymbol{L}_2}{\delta \boldsymbol{\omega}^{cd}} + \bar{\delta}_1 \boldsymbol{F} \wedge \frac{\delta \boldsymbol{L}_2}{\delta \boldsymbol{F}} \right). \tag{5.16}$$

Using now that $\frac{\partial \boldsymbol{d}*\boldsymbol{F}}{\partial \boldsymbol{de}^a} = \frac{\partial * \boldsymbol{F}}{\partial \boldsymbol{e}^a}$ and the explicit expressions for $\bar{\delta}_1 \boldsymbol{e}^a$, $\bar{\delta}_1 \bar{\boldsymbol{\psi}}_A$ et $\bar{\delta}_1 \boldsymbol{A}$ we get from the previous two equations:

$$\frac{\partial}{\partial \boldsymbol{de}^a} \left(\bar{\delta}_1 \boldsymbol{\omega}^{cd} \wedge \frac{\delta \boldsymbol{L}_2}{\delta \boldsymbol{\omega}^{cd}} + \bar{\delta}_1 \boldsymbol{F} \wedge \frac{\delta \boldsymbol{L}_2}{\delta \boldsymbol{F}} \right) = \frac{i\kappa}{2} \bar{\epsilon}_A \left(\gamma^5 \boldsymbol{F}_a + * \boldsymbol{F}_a \right) \wedge \varepsilon^A{}_B \boldsymbol{\psi}^B + \\ + \frac{i}{2} \bar{\epsilon}_A \gamma^5 \gamma_a \hat{\boldsymbol{D}} \boldsymbol{\psi}^A , \tag{5.17}$$

$$\frac{\partial}{\partial \boldsymbol{dA}} \left(\bar{\delta}_1 \boldsymbol{\omega}^{cd} \wedge \frac{\delta \boldsymbol{L}_2}{\delta \boldsymbol{\omega}^{cd}} + \bar{\delta}_1 \boldsymbol{F} \wedge \frac{\delta \boldsymbol{L}_2}{\delta \boldsymbol{F}} \right) = -i\kappa \bar{\epsilon}_A \varepsilon^A{}_B \gamma^5 \hat{\boldsymbol{D}} \boldsymbol{\psi}^B + \bar{\delta}_1 \boldsymbol{e}^a \wedge * \boldsymbol{F}_a - \\ - \bar{\delta}_1 \boldsymbol{e}^b \wedge \frac{\partial * \boldsymbol{F}}{\partial \boldsymbol{e}^b} . \tag{5.18}$$

Now the unknown quantities in the above equations are $\bar{\delta}_1 \boldsymbol{\omega}^{cd}$ and $\bar{\delta}_1 \boldsymbol{F}$. To find them, we can remark that the pair of equations (5.17)–(5.18) is a special case of (4.27), where now the variable x^α goes for $\boldsymbol{de}^a$ and $\boldsymbol{dA}$ and ω^i for $\boldsymbol{\omega}^{ab}$ and $\boldsymbol{F}$. As in the $\mathcal{N}_4 = 1$ case, we see from (5.17)–(5.18) that E_i and R_α are respectively linear and independent of x^α. We can then choose $\bar{\delta}_1 \omega^i$ to be independent of x^α (see discussion after equation (4.27)). In that case, the pair (5.17)–(5.18) becomes after some algebra:

$$-\frac{1}{4\kappa^2} \bar{\delta}_1 \boldsymbol{\omega}^{bc} \wedge \boldsymbol{\Sigma}_{abc} = \frac{i}{2} \bar{\epsilon}_A \gamma^5 \gamma_a \hat{\boldsymbol{D}} \boldsymbol{\psi}^A - \\ - \frac{i\kappa}{4} \bar{\epsilon}_A \left(F^{bc} - \gamma^5 * F^{bc} \right) \varepsilon^A{}_B \boldsymbol{\psi}^B \wedge \boldsymbol{\Sigma}_{abc} , \tag{5.19}$$

$$\bar{\delta}_1 \left(* \boldsymbol{F} \right) - \bar{\delta}_1 \boldsymbol{e}^a \wedge * \boldsymbol{F}_a = -i\kappa \bar{\epsilon}_A \varepsilon^A{}_B \gamma^5 \hat{\boldsymbol{D}} \boldsymbol{\psi}^B . \tag{5.20}$$

The last steep is to isolate $\bar{\delta}_1 \boldsymbol{\omega}^{ab}$ and $\bar{\delta}_1 \boldsymbol{F}$ from (5.19)–(5.20). The l.h.s. and the first term of the r.h.s. of (5.19) are analogous to the result (4.29) of $\mathcal{N}_4 = 1$ supergravity. Then it can be inverted using the matrix (3.3), giving three terms analogous to the Deser-Zumino ones. The second term of the r.h.s. of (5.19) is proportional to $\boldsymbol{\Sigma}_{abc}$ (as the l.h.s.) and thus does not require any more work:

$$\bar{\delta}_1 \boldsymbol{\omega}_{ab} = 2i\kappa^2 \bar{\epsilon} \gamma^5 \left(\boldsymbol{\gamma}_{(1)} * \hat{\psi}_{ab} - \frac{1}{2} \boldsymbol{e}_a \gamma^c * \hat{\psi}_{cb} + \frac{1}{2} \boldsymbol{e}_b \gamma^c * \hat{\psi}_{ca} \right) + \\ + i\kappa^3 \bar{\epsilon}_A \left(F_{ab} - \gamma^5 * F_{ab} \right) \varepsilon^A{}_B \boldsymbol{\psi}^B , \tag{5.21}$$

where $* \hat{\psi}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \hat{D}^\rho \psi^\sigma$ and $* F_{\mu\nu} := \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$.

To invert the second equation (5.20), we use the identity: $\delta \left(* \boldsymbol{F} \right) - \delta \boldsymbol{e}^a \wedge * \boldsymbol{F}_a = * \left(\delta \boldsymbol{F} - \delta \boldsymbol{e}^a \wedge \boldsymbol{F}_a \right)$, which can be proved straightforwardly, see also [18]. Then equation (5.20) implies that:

$$\bar{\delta}_1 \boldsymbol{F} = i\kappa \bar{\epsilon}_A \varepsilon^A{}_B \gamma^5 * \left(\hat{\boldsymbol{D}} \boldsymbol{\psi}^B \right) + \bar{\delta}_1 \boldsymbol{e}^a \wedge \boldsymbol{F}_a \tag{5.22}$$

Thus, the 1st order $\mathcal{N}_4 = 2$ lagrangean (5.1) is invariant under the supersymmetry transformation laws given by (5.11), (5.21) and (5.22).

It is interesting to note that using the new method it is possible to find a nice formula for the supersymmetry transformation laws, in the sense that they depend on the independent auxiliary fields (namely $\boldsymbol{\omega}^{ab}$ and $\boldsymbol{F}$) and not on the induced auxiliary fields (namely $\boldsymbol{\omega}_2^{ab}$ and $\boldsymbol{F}_2$ or $\boldsymbol{de}^a$ and $\boldsymbol{dA}$). This aesthetical criterion was the motivation for looking for a new method in section 4.3, and not simply stop with the general results (2.4), (3.6) and (3.8). In the next example, namely the $\mathcal{N}_{11} = 1$ supergravity, we will prove that such an "aesthetical" result does not exist.

6. 1st order formalism for $\mathcal{N}_{11} = 1$ supergravity

6.1 The 1st order lagrangean

The five basic fields of the 1st order eleven dimensional supergravity are the one-form vierbein $\boldsymbol{e}^a$, the so$(1,10)$ one-form (spin) connection $\boldsymbol{\omega}^a{}_b$, the Rarita-Schwinger one-form $\boldsymbol{\psi}$ and the three-form tensor $\boldsymbol{A} = \frac{1}{3!}A_{\mu\nu\rho}\boldsymbol{d}x^\mu\wedge\boldsymbol{d}x^\nu\wedge\boldsymbol{d}x^\rho$ and its associated field strength $\boldsymbol{F} = \frac{1}{4!}F_{\mu\nu\rho\sigma}\boldsymbol{d}x^\mu\wedge\boldsymbol{d}x^\nu\wedge\boldsymbol{d}x^\rho\wedge\boldsymbol{d}\sigma$. The eleven-form lagrangean can be rewritten as:[7]

$$\begin{aligned}\boldsymbol{L}_{11} = &-\frac{1}{4\kappa^2}\boldsymbol{R}^{ab}\wedge\boldsymbol{\Sigma}_{ab} + \frac{i}{2}\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(8)}\wedge\boldsymbol{D\psi} + \frac{i}{8}\left(\boldsymbol{\Theta}^a - \frac{i\kappa^2}{4}\bar{\boldsymbol{\psi}}\gamma^a\wedge\boldsymbol{\psi}\right)\wedge\boldsymbol{e}_a\wedge\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(6)}\wedge\boldsymbol{\psi} - \\ &-\frac{1}{2}\boldsymbol{F}\wedge{*\boldsymbol{F}} + ({*\boldsymbol{F}} + \boldsymbol{b})\wedge(\boldsymbol{dA} - \boldsymbol{a}) + \frac{1}{2}\boldsymbol{a}\wedge\boldsymbol{b} - \frac{\kappa}{3}\boldsymbol{A}\wedge\boldsymbol{dA}\wedge\boldsymbol{dA}\,, \end{aligned} \tag{6.1}$$

with the definitions:

$$\boldsymbol{a} := \frac{i\kappa}{4}\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(2)}\wedge\boldsymbol{\psi}\,, \qquad \boldsymbol{b} := \frac{i\kappa}{4}\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(5)}\wedge\boldsymbol{\psi}\,. \tag{6.2}$$

We also defined $\boldsymbol{\gamma}_{(n)} := \frac{1}{n!}\boldsymbol{\gamma}_{(1)}\wedge\boldsymbol{\gamma}_{(1)}\wedge\ldots\wedge\boldsymbol{\gamma}_{(1)}$ (n times). The notations and conventions are the same as in the previous subsections, generalized straightforwardly

[7]Note that (6.1) (whose gravitational part was first proposed in [9]) differs slightly from the original formulation of eleven dimensional supergravity of [24]: In fact (6.1) is just the standard lagrangean reexpressed in terms of the new connection $\boldsymbol{\omega}^a{}_b$. This connection, as an independent variable, is *not* the independent connection of [24] but is shifted in such a way that its induced value is what was called in that paper $\hat{\boldsymbol{\omega}}^a{}_b$. In this form, the equations of motion of the connection (6.6) are automatically supercovariant, in the sense that their supersymmetry variation (6.10)–(6.12) does not contain the derivative of the gauge parameter. Finally, $\boldsymbol{L}_{11}$ of (6.1) is a complete 1st order formulation, with the four form also treated as an independent field. We should emphasize that the use of a supercovariant connection is a technical help not a fundamental requirement in this work.

to eleven dimensions.[8] The equations of motion for (6.1) are given by:

$$
\begin{aligned}
\frac{\delta \boldsymbol{L}_{11}}{\delta \boldsymbol{e}^a} &= -\frac{1}{4\kappa^2}\boldsymbol{R}^{bc}\wedge\boldsymbol{\Sigma}_{bca} - \frac{i}{4}\bar{\boldsymbol{\psi}}\wedge(2\boldsymbol{\gamma}_{(7)a} + \boldsymbol{e}_a\wedge\boldsymbol{\gamma}_{(6)})\wedge\boldsymbol{D\psi} + \\
&\quad + \frac{i}{4}\boldsymbol{\Theta}^b\wedge\bar{\boldsymbol{\psi}}\wedge\left(\boldsymbol{\gamma}_{(6)}\eta_{ab} + \boldsymbol{\gamma}_{(5)(a}\wedge\boldsymbol{e}_{b)}\right)\wedge\boldsymbol{\psi} + \boldsymbol{\tau}_a = 0\,, && (6.3)\\
\frac{\delta \boldsymbol{L}_{11}}{\delta \bar{\boldsymbol{\psi}}} &= i\boldsymbol{\gamma}_{(8)}\wedge\hat{\boldsymbol{D}}\boldsymbol{\psi} - \frac{i\kappa}{2}\boldsymbol{\gamma}_{(5)}\wedge\boldsymbol{\psi}\wedge(\boldsymbol{F} - \boldsymbol{dA} + \boldsymbol{a}) + \\
&\quad + \frac{i}{4}\left(\boldsymbol{\Theta}^a - \frac{i\kappa^2}{2}\bar{\boldsymbol{\psi}}\gamma^a\wedge\boldsymbol{\psi}\right)\wedge\left(2\boldsymbol{\gamma}_{(7)a} - \boldsymbol{\gamma}_{(6)}\wedge\boldsymbol{e}_a\right)\wedge\boldsymbol{\psi} = 0\,, && (6.4)\\
\frac{\delta \boldsymbol{L}_{11}}{\delta \boldsymbol{A}} &= \boldsymbol{d}(*\boldsymbol{F} + \boldsymbol{b}) - \kappa\boldsymbol{dA}\wedge\boldsymbol{dA} = 0\,, && (6.5)\\
\frac{\delta \boldsymbol{L}_{11}}{\delta \boldsymbol{\omega}^{ab}} &= -\frac{1}{4\kappa^2}\left(\boldsymbol{\Theta}^c - \frac{i\kappa^2}{2}\bar{\boldsymbol{\psi}}\gamma^c\wedge\boldsymbol{\psi}\right)\wedge\boldsymbol{\Sigma}_{abc} = 0\,, && (6.6)\\
\frac{\delta \boldsymbol{L}_{11}}{\delta \boldsymbol{F}} &= *(\boldsymbol{dA} - \boldsymbol{a} - \boldsymbol{F}) = 0\,, && (6.7)
\end{aligned}
$$

where we also used the shorthand notation defined by (A.3).

Some additional comments are needed:

- The lagrangean (6.1) can be rewritten (up to a surface term) in the following suggestive form:

$$
\begin{aligned}
\boldsymbol{L}_{11} &= -\frac{1}{4\kappa^2}\left[\frac{1}{2}\#\boldsymbol{\omega}_a\wedge\boldsymbol{\omega}^a + (\#\boldsymbol{\omega}_a + \boldsymbol{\beta}_a)\wedge(\boldsymbol{de}^a - \boldsymbol{\alpha}^a) + \frac{1}{2}\boldsymbol{\alpha}^a\wedge\boldsymbol{\beta}_a\right] + \\
&\quad + \frac{i}{2}\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(8)}\wedge\boldsymbol{d\psi} - \frac{1}{2}\boldsymbol{F}\wedge*\boldsymbol{F} + (*\boldsymbol{F} + \boldsymbol{b})\wedge(\boldsymbol{dA} - \boldsymbol{a}) + \\
&\quad + \frac{1}{2}\boldsymbol{a}\wedge\boldsymbol{b} - \frac{\kappa}{3}\boldsymbol{A}\wedge\boldsymbol{dA}\wedge\boldsymbol{dA}\,, && (6.8)
\end{aligned}
$$

  where we defined $\boldsymbol{\omega}^a := \boldsymbol{\omega}^a{}_b\wedge\boldsymbol{e}^b$, $\#\boldsymbol{\omega}_a := \boldsymbol{\omega}^{bc}\wedge\boldsymbol{\Sigma}_{abc}$ and

$$
\boldsymbol{\alpha}^a := \frac{i\kappa^2}{2}\bar{\boldsymbol{\psi}}\gamma^a\wedge\boldsymbol{\psi}\,, \qquad \boldsymbol{\beta}_a := -\frac{i\kappa^2}{2}\boldsymbol{e}_a\wedge\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(6)}\wedge\boldsymbol{\psi}\,. \qquad (6.9)
$$

  Note that the # operation is up to a trace, a Hodge-duality operator: from the definitions, $\#\boldsymbol{\omega}_a = *(\boldsymbol{\omega}_a + \boldsymbol{e}^b\wedge\boldsymbol{i}_a\boldsymbol{\omega}_b - 2\boldsymbol{e}_a\wedge\boldsymbol{i}_b\boldsymbol{\omega}^b)$.

- The energy-momentum tensor of (6.3) is given by

$$
\begin{aligned}
\boldsymbol{\tau}_a &= \frac{\partial}{\partial \boldsymbol{e}^a}(*\boldsymbol{F} + \boldsymbol{b})\wedge\boldsymbol{dA} + \\
&\quad + \frac{\partial}{\partial \boldsymbol{e}^a}\left(-*\boldsymbol{F}\wedge\boldsymbol{a} - \frac{1}{2}\boldsymbol{F}\wedge*\boldsymbol{F} + \frac{\kappa^2}{16}\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(1)}\wedge\boldsymbol{\psi}\wedge\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(6)}\wedge\boldsymbol{\psi}\right).
\end{aligned}
$$

[8] Namely $\gamma^0\gamma^1\ldots\gamma^{10} = 1 = \varepsilon_{01\cdots10}$ and $\eta^{ab} = \{-,+,\cdots,+\}$, see also appendix A.

- The "supercovariant-hatted" derivative was derived in [24] and is given by: $\hat{\boldsymbol{D}}\boldsymbol{\psi} := \boldsymbol{D}\boldsymbol{\psi} - \frac{\kappa}{12^2}\left(\gamma_{abcde} + 8\gamma_{bcd}\eta_{ea}\right)\boldsymbol{e}^a F^{bcde}\wedge\boldsymbol{\psi}$. The equation of motion (6.4) can be computed only after making use of

  (i) a Fierz rearrangement:

$$\begin{aligned} &4\boldsymbol{\gamma}_{(7)a}\wedge\boldsymbol{\psi}\wedge(\bar{\boldsymbol{\psi}}\gamma^a\wedge\boldsymbol{\psi}) + \\ &+\boldsymbol{\gamma}_{(1)}\wedge\boldsymbol{\psi}\wedge(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(6)}\wedge\boldsymbol{\psi}) - \boldsymbol{\gamma}_{(6)}\wedge\boldsymbol{\psi}\wedge(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(1)}\wedge\boldsymbol{\psi}) + \\ &+\boldsymbol{\gamma}_{(5)}\wedge\boldsymbol{\psi}\wedge(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(2)}\wedge\boldsymbol{\psi}) - \boldsymbol{\gamma}_{(2)}\wedge\boldsymbol{\psi}\wedge(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(5)}\wedge\boldsymbol{\psi}) = 0\,; \end{aligned}$$

  (ii) a supercovariant derivative identity:

$$i\boldsymbol{\gamma}_8\wedge\hat{\boldsymbol{D}}\boldsymbol{\psi} = i\boldsymbol{\gamma}_8\wedge\boldsymbol{D}\boldsymbol{\psi} + \frac{i\kappa}{2}\left(\boldsymbol{\gamma}_{(5)}\wedge\boldsymbol{F} + \boldsymbol{\gamma}_{(2)}\wedge*\boldsymbol{F}\right)\wedge\boldsymbol{\psi}\,.$$

  As usual, the last two terms in the r.h.s. of (6.4) vanish if the equations of motion of the auxiliary fields $\boldsymbol{\omega}^{ab}$ and $\boldsymbol{F}$ are used.

- We fixed the action in such a way that the induced auxiliary fields $\boldsymbol{\omega}_2^{ab}$ and $\boldsymbol{F}_2$ (computed as the only solutions of (6.6)–(6.7)) are supercovariants, in the sense that their supersymmetry transformation laws (induced by (6.10)–(6.12)) do not contain the derivatives of the gauge parameter $\bar{\epsilon}$. Note however that we paid a price for this, namely we changed the quartic fermionic terms which seem unavoidable beyond 4 dimensions for pure supergravities. In the previous example the graviphoton did also cause such a complication.

6.2 The supersymmetry transformation law

The lagrangean (6.1) is invariant in a 2nd order sense (ie after using freely the auxiliary fields equations of motion) under the following supersymmetry transformation laws:

$$\delta_2\bar{\boldsymbol{\psi}} = \hat{\boldsymbol{D}}_2\bar{\epsilon} := \boldsymbol{D}_2\bar{\epsilon} + \frac{\kappa}{12^2}\bar{\epsilon}\left(\gamma_{abcde} - 8\gamma_{bcd}\eta_{ea}\right)\boldsymbol{e}^a F_2^{bcde}\,, \tag{6.10}$$

$$\delta_2\boldsymbol{e}^a = i\kappa^2\bar{\epsilon}\gamma^a\boldsymbol{\psi}\,, \tag{6.11}$$

$$\delta_2\boldsymbol{A} = \frac{i\kappa}{2}\bar{\epsilon}\boldsymbol{\gamma}_{(2)}\wedge\boldsymbol{\psi}\,. \tag{6.12}$$

As before, a 1st order formulation of eleven dimensional supergravity is given by equations (6.10)–(6.12) (following (2.4)), and by equations (3.6) and (3.8) for the auxiliary fields. In a more explicit way, these are:

$$\begin{aligned} \delta_1\boldsymbol{\omega}_{ab} = &-i\kappa^2\bar{\epsilon}\left(\gamma_c\hat{\psi}_{2ab} + \gamma_a\hat{\psi}_{2cb} - \gamma_b\hat{\psi}_{2ca}\right)\boldsymbol{e}^c + \frac{1}{2}\left(T_{a,bc} - T_{c,ab} + T_{b,ca}\right)\boldsymbol{e}^c + \\ &+\frac{i\kappa^3}{72}\bar{\epsilon}\left(\gamma_{abc_1\ldots c_4} + 24\eta_{ac_1}\eta_{bc_2}\gamma_{c_3c_4}\right)F_2^{c_1\cdots c_4}\boldsymbol{\psi}\,, \end{aligned} \tag{6.13}$$

where $\hat{\psi}_{2ab} := \hat{D}_{2[a}\psi_{b]}$ and $\hat{\boldsymbol{D}}_2$ is the 2^{nd} order supercovariant derivative constructed with the induced auxiliary fields, namely $\boldsymbol{\omega}_2^{ab}$ and $\boldsymbol{F}_2$, see (6.10). Moreover, the tensor $T_{a,bc}$ (which vanishes modulo the equation of motion of the connection) was defined just after equation (3.6).

The transformation law for the field strength can also be computed following equation (3.8). The final result is:[9]

$$\delta_1 \boldsymbol{F} = \frac{i\kappa}{2}\bar{\epsilon}\boldsymbol{\gamma}_{(2)}\wedge\hat{\boldsymbol{D}}_2\boldsymbol{\psi} + \delta_2\boldsymbol{e}^a\wedge\boldsymbol{i}_a\boldsymbol{F} - \frac{\delta_2 e}{2e}\Delta\boldsymbol{F}\,, \tag{6.14}$$

with e the determinant of the vielbein.

A 1^{st} order formulation for the gravitational part of eleven-dimensional supergravity was first derived in [9], it did not treat however the four form field strength as an independent field. It differs from (6.10)–(6.13) by some trivial gauge transformation of the type (3.12) (to see how to proceed, it is enough to follow the method given in section 4.2). We could also add another trivial gauge transformation to replace the $\boldsymbol{F}_2$ by $\boldsymbol{F}$ in (6.10), and then $\hat{\boldsymbol{D}}_2$ by $\hat{\boldsymbol{D}}$. The net result would be to add a contribution to (6.14) proportional to the equations of motion of the gravitino.

It is not obvious at all that the final expressions for $\delta_1\boldsymbol{\omega}^{ab}$ and $\delta_1\boldsymbol{F}$ would be simpler or "more geometrical" after adding these trivial gauge transformations. The question we would like to answer is thus to know whether there exists some appropriate trivial gauge transformation, which, added to (6.10)–(6.14), gives as "nice" a result as in the $\mathcal{N}_4 = 2$ supergravity case (5.21)–(5.22).

By "nice" we mean the following: we know (from the new method of section 4.3) that there exist transformation laws $\bar{\delta}_1\boldsymbol{\omega}^{ab}$ and $\bar{\delta}_1\boldsymbol{F}$ which, together with

$$\begin{aligned} \bar{\delta}_1\bar{\boldsymbol{\psi}} &:= \hat{\boldsymbol{D}}\bar{\epsilon}\,, \\ \bar{\delta}_1\boldsymbol{e}^a &:= \delta_2\boldsymbol{e}^a\,, \\ \bar{\delta}_1\boldsymbol{A} &:= \delta_2\boldsymbol{A} \end{aligned} \tag{6.15}$$

leave the lagrangean (6.1) invariant.

A "nice" formula would be one where these $\bar{\delta}_1\boldsymbol{\omega}^{ab}$ and $\bar{\delta}_1\boldsymbol{F}$ would be independent of $\boldsymbol{de}^a$ (ie independent of $\Delta\boldsymbol{\omega}^{ab}$) and of $\boldsymbol{dA}$ (ie independent of $\Delta\boldsymbol{F}$) respectively. We will end this section by giving strong arguments supporting the idea that such a "nice" formula does not exist.

As in previous examples, the $(D-1)$-form defined by (4.18) is given by formula (4.22).[10] Now, using the gravitino's equation of motion (6.4), we can compute

[9] To find the result of eq. (6.14), we need the following eleven-dimensional Fierz identity: $\gamma^a\boldsymbol{\psi}\wedge\bar{\boldsymbol{\psi}}\wedge\gamma_{(1)a}\wedge\boldsymbol{\psi} + \gamma_{(1)a}\wedge\boldsymbol{\psi}\wedge\bar{\boldsymbol{\psi}}\gamma^a\wedge\boldsymbol{\psi} = 0$, which can be proved using the general formula (A.6) with $\boldsymbol{\psi}_1 = \boldsymbol{\psi}$, $\bar{\boldsymbol{\psi}}_2 = \bar{\boldsymbol{\psi}}$ and $\boldsymbol{\psi}_3 = \gamma_{(1)a}\wedge\boldsymbol{\psi}$ for the first term and $\boldsymbol{\psi}_1 = \boldsymbol{\psi}$, $\bar{\boldsymbol{\psi}}_2 = \bar{\boldsymbol{\psi}}$ and $\boldsymbol{\psi}_3 = \gamma_a\boldsymbol{\psi}$ for the second one.

[10] Remember that if the induced auxiliary fields are supercovariants, so are the auxiliary fields themselves, and then, only the gravitino transformation law contains $\boldsymbol{d}\bar{\epsilon}$-terms.

the eleven dimensional version of the pair of equations (5.13)–(5.14):

$$\begin{aligned}\frac{\delta \boldsymbol{W}_{\bar\epsilon}}{\delta \boldsymbol{e}^a} &= \frac{i}{4}\bar\epsilon\left(2\boldsymbol{\gamma}_{(7)a} + \boldsymbol{e}_a\wedge\boldsymbol{\gamma}_{(6)}\right)\wedge\boldsymbol{D\psi} + \frac{i}{4}\boldsymbol{D}\bar\epsilon\left(2\boldsymbol{\gamma}_{(7)a} - \boldsymbol{e}_a\wedge\boldsymbol{\gamma}_{(6)}\right)\wedge\boldsymbol{\psi} - \\ &\quad - \frac{i}{2}\boldsymbol{\Theta}^b\bar\epsilon\left(\eta_{ab}\boldsymbol{\gamma}_{(6)} + \boldsymbol{\gamma}_{(5)(a}\wedge\boldsymbol{e}_{b)}\right) + \frac{i\kappa}{2}\bar\epsilon\boldsymbol{\gamma}_{(4)a}\wedge\boldsymbol{\psi}\wedge\boldsymbol{dA} + \text{more} \qquad (6.16)\\ \frac{\delta \boldsymbol{W}_{\bar\epsilon}}{\delta \boldsymbol{A}} &= \frac{i\kappa}{2}\boldsymbol{D}\left(\bar\epsilon\boldsymbol{\gamma}_{(5)}\wedge\boldsymbol{\psi}\right), \qquad (6.17)\end{aligned}$$

where "more" goes for terms proportional to $\bar\epsilon\boldsymbol{\psi}^3$ and $\bar\epsilon\boldsymbol{\psi}\boldsymbol{F}$ and useless for our purpose.

Using the identity (4.19), the equations (6.16)–(6.17) are respectively equal to:

$$\frac{i}{4}\bar\delta_1\bar{\boldsymbol{\psi}}\wedge\left(2\boldsymbol{\gamma}_{(7)a} - \boldsymbol{e}_a\wedge\boldsymbol{\gamma}_{(6)}\right)\boldsymbol{\psi} + \text{more} + \frac{\partial}{\partial \boldsymbol{de}^a}\left(\bar\delta_1\boldsymbol{\omega}^{cd}\wedge\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{\omega}^{cd}} + \bar\delta_1\boldsymbol{F}\wedge\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{F}}\right) \qquad (6.18)$$

$$\frac{i\kappa}{2}\bar\delta_1\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\gamma}_{(5)}\wedge\boldsymbol{\psi} - 2\kappa\bar\delta_1\boldsymbol{A}\wedge\boldsymbol{dA} + \text{more} + \frac{\partial}{\partial \boldsymbol{dA}}\left(\bar\delta_1\boldsymbol{\omega}^{cd}\wedge\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{\omega}^{cd}} + \bar\delta_1\boldsymbol{F}\wedge\frac{\delta\boldsymbol{L}_2}{\delta\boldsymbol{F}}\right) \qquad (6.19)$$

and "more" has the same meaning as before.

The above two equalities are a special case of the general equation (4.27), where x^α goes for $\boldsymbol{de}^a$ and $\boldsymbol{dA}$ and ω^i for $\boldsymbol{\omega}^{ab}$ and $\boldsymbol{F}$. As before, the auxiliary field equations of motion (6.6)–(6.7) E_i are linear in x^α (namely in $\boldsymbol{de}^a$ and $\boldsymbol{dA}$), ie $E_i = E_{i\alpha}x^\alpha + E_{i0}$. The very big difference with the four dimensional supergravities is that now R_α is also *linear* in (and *not* independent of) x^α (by rearranging of equations (6.16)–(6.19)), $R_\alpha = R_{\alpha\beta}x^\beta + R_{\alpha 0}$. That means that $\bar\delta_1\omega^i$ cannot be independent of x^α. Instead, we can choose it to be linear in this variable x^α. The practical reasons for that annoying feature are the gammalogy beyond four dimensions and also the presence of the Chern-Simons term.

This means in particular that $\bar\delta_1\boldsymbol{\omega}^{ab}$ are linear in $\boldsymbol{de}^a$, and then in $\boldsymbol{\omega}_2^{ab}$ (it is always possible to rewrite the derivative of the vierbein in term of the induced connection). So it will be impossible to rewrite $\bar\delta_1\boldsymbol{\omega}^{ab}$ in a "nice" way (nicer than the straightforward result given by (6.10)–(6.14)), in the sense that only the independent auxiliary field appears in the transformation laws and not the induced one. The same is true for $\boldsymbol{F}$ and $\boldsymbol{dA}$.

This problem seems to appear as soon as we go beyond 4 dimensions. The reason for that remains however unclear for us. On the other hand this may have some importance for the construction of superspace versions of these theories. A first attempt at a deeper understanding should address the 5 dimensional case where auxiliary fields are in principle known.

For completeness we should also mention related works by Bars, MacDowell, Higuchi and Kallosh [25] where the (partially) first order formalism of [9] is combined with an identification of the three form gauge field with supertorsion so as to lead to more equivalent forms of classical 11 dimensional supergravity. Recently a preprint [8] appeared where some first order results for the three form sector of 11d supergravity are listed, they should be obtainable from our formulas by elimination of the Lorentz connection.

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A. Conventions and useful formulas

The conventions are the following for a D-dimensional spacetime dimension (see [26] for a more complete discussion): $\eta_{ab} = \{-,+,\ldots,+\}$, $\varepsilon_{01\ldots(D-1)} = 1$, γ_a ($a = 0,\ldots,D-1$) are D gamma matrices of dimension $2^{[D/2]}$ which satisfy the Clifford algebra $\{\gamma_a,\gamma_b\} = 2\eta_{ab}$. Here, $[r]$ is the integer part of r (i.e. [4.5]=4). If D is even (odd), $\gamma^{D+1}(\mathbb{1}) := \gamma^0\ldots\gamma^{D-1}$, $\left(\gamma^{D+1}\right)^2 = (-)^{\frac{D(D-1)}{2}+1}\mathbb{1}$.

Some definitions and useful γ matrix product formulas are:

$$\gamma_{a_1\cdots a_n} := \gamma_{[a_1}\cdots\gamma_{a_n]} \tag{A.1}$$

we define by $\boldsymbol{\gamma}_{(n)}$ the (n)-form:

$$\boldsymbol{\gamma}_{(n)} := \frac{\gamma_{a_1\cdots a_n}}{n!}\boldsymbol{e}^{a_1}\wedge\cdots\wedge\boldsymbol{e}^{a_n} \tag{A.2}$$

and also

$$\boldsymbol{\gamma}_{(n)a_1\cdots a_m} := \boldsymbol{i}_{a_1}\cdots\boldsymbol{i}_{a_m}\boldsymbol{\gamma}_{(n+m)} := \gamma_{a_1\cdots a_m b_1\cdots b_n}\frac{\boldsymbol{e}^{b_1}\wedge\cdots\wedge\boldsymbol{e}^{b_n}}{n!} \tag{A.3}$$

for any integers m and n, $\boldsymbol{i}_a\ldots$ mean contraction with the relevant frame vector.

Now, from the standard decomposition formula (complete antisymmetrisation in $a_m\cdots a_1$ and $b_1\cdots b_n$ is understood in the above formulas):

$$\gamma_{a_m\ldots a_1}\gamma_{b_1\ldots b_n} = \sum_{i=0}^{\min(m,n)} i!\binom{m}{i}\binom{n}{i}\eta_{a_1b_1}\ldots\eta_{a_ib_i}\gamma_{a_m\ldots a_{i+1}b_{i+1}\ldots b_n} \tag{A.4}$$

it is easy to derive:

$$\begin{aligned}
\boldsymbol{\gamma}_{(m)}\wedge\boldsymbol{\gamma}_{(n)} &= \binom{m+n}{n}\boldsymbol{\gamma}_{(m+n)}\,,\\
\gamma_{a_m\cdots a_1}\boldsymbol{\gamma}_{(n)} &= \sum_{i=0}^{\min(m,n)}\binom{m}{i}\boldsymbol{e}_{a_1}\wedge\cdots\wedge\boldsymbol{e}_{a_i}\wedge\boldsymbol{\gamma}_{(n-i)a_m\cdots a_{i+1}}\,,\\
(-)^{m.n}\boldsymbol{\gamma}_{(n)}\gamma_{a_m\cdots a_1} &= \sum_{i=0}^{\min(m,n)}\binom{m}{i}(-)^i\boldsymbol{e}_{a_1}\wedge\cdots\wedge\boldsymbol{e}_{a_i}\wedge\boldsymbol{\gamma}_{(n-i)a_m\cdots a_{i+1}}\,,
\end{aligned} \tag{A.5}$$

where $\binom{m}{i}$ stands for $\frac{m!}{(m-i)!i!}$.

The general Fierz formula in any spacetime dimension D is given by:

$$\boldsymbol{\psi}_1 \wedge \left(\bar{\boldsymbol{\psi}}_2 \wedge \boldsymbol{\psi}_3\right) = \frac{(-)^{1+p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1}}{2^{[\frac{D+1}{2}]}} \sum_{n=0}^{D} \frac{1}{n!} \gamma^{a_n \ldots a_1} \boldsymbol{\psi}_3 \wedge \left(\bar{\boldsymbol{\psi}}_2 \wedge \gamma_{a_1 \ldots a_n} \boldsymbol{\psi}_1\right), \quad \text{(A.6)}$$

where $\boldsymbol{\psi}_i$ is a (p_i)-form anticommuting spinor ($i = 1, \ldots, 3$). Note that the summation in (A.6) contains $(D+1)$ terms. In the case where D is odd, half of them can be eliminated by a Hodge-duality transformation (recall that in that case $\gamma^{D+1} \sim \mathbb{1}$). Then the summation goes only up to $\frac{D-1}{2}$ and a factor of 2 automatically cancels out in the denominator of (A.6).

The Majorana (form)-spinors (which exist in $D = 4$ and $D = 11$) satisfy the symmetry property:

$$\bar{\boldsymbol{\psi}}_1 \wedge \gamma^{a_1 \cdots a_n} \boldsymbol{\psi}_2 = (-)^{p_1 \cdot p_2 + \frac{n(n+1)}{2}} \bar{\boldsymbol{\psi}}_2 \wedge \gamma^{a_1 \cdots a_n} \boldsymbol{\psi}_1 \quad \text{(A.7)}$$

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